Continuous formulation of the Loop Quantum Gravity phase space

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In this paper, we study the discrete classical phase space of loop gravity, which is expressed in terms of the holonomy-flux variables, and show how it is related to the continuous phase space of general relativity. In particular, we prove an isomorphism between the loop gravity discrete phase space and the symplectic reduction of the continuous phase space with respect to a flatness constraint. This gives for the first time a precise relationship between the continuum and holonomy-flux variables. Our construction shows that the fluxes depend on the three-geometry, but also explicitly on the connection, explaining their non commutativity. It also clearly shows that the flux variables do not label a unique geometry, but rather a class of gauge-equivalent geometries. This allows us to resolve the tension between the loop gravity geometrical interpretation in terms of singular geometry, and the spin foam interpretation in terms of piecewise flat geometry, since we establish that both geometries belong to the same equivalence class. This finally gives us a clear understanding of the relationship between the piecewise flat spin foam geometries and Regge geometries, which are only piecewise-linear flat: While Regge geometry corresponds to metrics whose curvature is concentrated around straight edges, the loop gravity geometry correspond to metrics whose curvature is concentrated around not necessarily straight edges.

Introduction

The classical starting point of Loop Quantum Gravity (LQG) [1, 2] is a Hamiltonian formulation of general relativity in terms of first order connection and triad variables. The basic fields parametrizing the phase space are chosen to be the $\mathfrak{su}(2)$-valued Ashtekar-Barbero connection A [3], and its canonically conjugate densitized triad field E, both being defined over spatial hypersurfaces foliating the spacetime manifold. The theory comes with a set of first class constraints, namely, the vector constraint generating diffeomorphisms of the spatial hypersurface, the scalar constraint generating time reparametrizations, and the Gauss constraint generating internal SU(2) gauge transformations.

As a first step towards the construction of the quantum theory, one defines a smearing of the classical Poisson algebra formed by the canonical pair (A, E) by introducing oriented graphs. Given a graph Γ embedded in the spatial manifold, the continuous variables $A(x)$ and $E(x)$ are replaced by a pair (h_e, X_e) associated to each edge e. The variable $h_e \in \mathrm{SU}(2)$ corresponds to the holonomy of the connection along the edge e, and $X_e \in \mathfrak{su}(2)$ represents the "electric" flux of the densitized triad field across a surface dual to e. At the quantum level these new variables form the so-called holonomy-flux algebra [4], which is a cornerstone of the entire construction of LQG. The Hilbert space $\mathcal{H}_\Gamma$ of representations associated with this algebra is the so-called spin network Hilbert space. It captures only a finite number of degrees of freedom in the theory. One recovers

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the continuous kinematical Hilbert space by taking the projective limit of graph Hilbert spaces $\mathcal{H}_\Gamma$. The main challenge is then to formulate a consistent and semi-classically meaningful version of the Hamiltonian constraint acting on the spin network basis.

In this construction, two very different procedures are realized at once. There is a discretization procedure in which the continuous fields are replaced by discrete holonomies and fluxes associated with graphs, and in the same stroke, these variables are promoted into quantum operators. The main idea we want to take advantage of is that the processes of discretization and quantization are totally independent, a point that has been unappreciated until now. In this work we would like to disentangle these two steps. We propose to study only the process of discretization using graphs, without delving into the quantization of the theory. This means that we first associate to a given graph a finite-dimensional holonomy-flux phase space generated by $(h_e, X_e) \in T^*\mathrm{SU}(2)$. The phase space of loop gravity on a graph is obtained as a direct product over the edges of $\mathrm{SU}(2)$ cotangent bundles. The main point of the present paper is to understand the exact relationship between this finite-dimensional discrete phase space, and the continuous phase space of gravity. We show explicitly that an element of the discrete phase space represents a specific equivalence class of continuous geometries.

The advantage of considering classical loop gravity is threefold. First, it provides a truncation of the classical phase space of gravity in terms of finite-dimensional holonomy-flux phase spaces, whose quantizations are given by spin network states. Second, it allows us to shed some light on the geometrical interpretation of the holonomy-flux variables, and the type of geometry that they represent. For instance, we will see that both the singular geometry of LQG and the piecewise flat geometry of spin foam models are represented by the same flux data as two representatives of the same equivalence class. As we will see in the end, our result also allows us to understand more precisely the relationship between the spin foam geometrical interpretation and Regge geometry. Namely, it shows that twisted geometries [5] described by fluxes can be understood as piecewise flat geometries which are not necessarily piecewise-linear flat, as is the case for Regge geometry [6, 7]. Finally, this approach is designed to allow us address at the classical level one of the most challenging questions of LQG: Is it possible to express a proper gravitational dynamics in terms of holonomies and fluxes? If there is a clear positive answer to this question at the classical level, then the quantization of loop gravity will be reduced to the treatment of quantization ambiguities in a finite-dimensional system. If, on the other hand, we get a negative answer at the classical level, then no quantization in terms of holonomy-flux variables can express the quantum gravitational dynamics. It is therefore of utmost importance to eventually understand the classical dynamics of general relativity in terms of the holonomy-flux representation.

Let us stress that the classical picture of the loop gravity phase space that we develop here is, when quantized, related to the picture first proposed by Bianchi in [8]. In this precursor work, it is argued that the spin network Hilbert space can be identified with the state space of a topological theory on a flat manifold with defects. Our analysis makes the same type of identification at the classical level and emphasizes the fact that the frame field determines only an equivalence class of geometries. The idea that the discrete data labels only an equivalence class of geometries has already been advocated in [9] on a general basis. Our approach gives a precise understanding of which set or equivalence class of continuous geometries is represented by the discrete geometrical data.

We begin in section I by defining the continuous phase space of gravity in terms of the connection and triad variables A and E, and recall some facts about the process of symplectic reduction. In section II we introduce the discrete classical spin network phase space associated to a graph. In particular, we explain how to obtain the discrete data (h_e, X_e) starting from the continuous fields A and E, and showing the fluxes cannot depend only on E but need to involve the connection in their definition. This construction explains why the flux variable carries information about

both intrinsic and extrinsic geometry, in agreement with what has been pointed out already in [5]. In section III, we prove that the discrete holonomy-flux phase space can be obtained as a symplectic reduction of the continuous phase space. This shows that the discrete data corresponds to an equivalence class of continuous three-geometries related by gauge transformations. In section IV we show that given a particular gauge choice, the discrete data can be used to reconstruct a configuration of the continuous fields. We will show in particular that it is possible to represent a given equivalence class of geometries by either a singular gauge choice in agreement with the LQG interpretation of polymer geometry, or a flat gauge choice corresponding to the geometrical interpretation of spin foams. Finally, in section V we discuss the notion of cylindrical consistency and cylindrical operators, and explain how it is possible to relate operators constructed on the discrete and the continuous phase spaces.

Notations are such that $\mu,\nu,\dots$ refer to spacetime indices, $a,b,\dots$ to spatial indices, $I,J,\dots$ to Lorentzian indices, and $i,j,\dots$ to $\mathfrak{su}(2)$ indices. We will assume that the four-dimensional spacetime manifold is topologically $\Sigma\times\mathbb{R}$, where Σ is a three-dimensional manifold without boundaries.

I. CONTINUOUS PHASE SPACE OF GRAVITY

The loop approach to quantum gravity relies on the well-known idea that the phase space of Lorentzian or Riemannian general relativity can be parametrized in terms of an $\mathfrak{su}(2)$-valued connection one-form A^i_a and a densitized triad field $\tilde{E}^a_i$, both fields being defined over a base three-dimensional spacetime manifold Σ (which here we assume to be isomorphic to $\mathbb{R}^3$). The $\mathfrak{su}(2)$ Ashtekar-Barbero connection A^i_a is related to the spacetime $\mathfrak{so}(3,1)$ spin connection ω^{IJ}_μ and to the geometrodynamical variables of the ADM phase space via

$$A^i_a\equiv\frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}_a+\gamma\omega^{0i}_a=\Gamma^i_a+\gamma K^i_a, \tag{1.1}$$

where $\gamma\in\mathbb{R}-\{0\}$ is the Barbero-Immirzi parameter, Γ^i_a is the Levi-Civita spin connection, and K^i_a the extrinsic curvature one-form. The densitized triad and the three-dimensional frame field e^i_a are related by

$$\tilde{E}^a_i=\frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_be^k_c,\qquad\qquad \det(e)e^i_a=\frac{1}{2}\epsilon^{ijk}\epsilon_{abc}\tilde{E}^b_j\tilde{E}^c_k. \tag{1.2}$$

These variables form the Poisson algebra

$$\big\{A^i_a(x),A^j_b(y)\big\}=\big\{\tilde{E}^a_i(x),\tilde{E}^b_j(y)\big\}=0,\qquad\qquad \big\{A^i_a(x),\tilde{E}^b_j(y)\big\}=\gamma\delta^i_j\delta^b_a\delta^3(x-y). \tag{1.3}$$

The classical configuration space of the theory is the space $\mathcal{A}$ of smooth connections on Σ. The phase space is the cotangent bundle $\mathcal{P}\equiv T^*\mathcal{A}$, and carries a natural symplectic potential. In the following we will denote by $E_{ab\,i}$ (without tilde) the Lie algebra-valued two-form related to the densitized vector $\tilde{E}^i_a$ through

$$E_{ab\,i}\equiv\epsilon_{abc}\tilde{E}^c_i,\qquad\qquad E_i\equiv E_{ab\,i}\mathrm{d}x^a\wedge\mathrm{d}x^b. \tag{1.4}$$

The symplectic potential of the cotangent bundle is given by

$$\Theta=\int_\Sigma E_i\wedge\delta A^i=\int_\Sigma\mathrm{Tr}\left(E\wedge\delta A\right), \tag{1.5}$$

where we denote by Tr the natural metric on $\mathfrak{su}(2)$ which is invariant under the adjoint action

$\mathrm{Ad}_{\mathrm{SU}(2)}$ of the group. The phase space $\mathcal{P}$ also carries an action of the gauge group SU(2) and of spatial diffeomorphisms. In fact, since $\mathcal{P}$ is $18\cdot\infty^3$-dimensional, the (first class) constraints of the canonical theory have to be taken into account in order to obtain the physical phase space with $4\cdot\infty^3$ degrees of freedom. This can be achieved through the process of symplectic (or Hamiltonian) reduction, which we now describe.

Let P be a symplectic manifold, which is seen as the classical phase space of the theory of interest, and G a group of transformations. Suppose that the infinitesimal group transformations are generated via Poisson bracket by a Hamiltonian H. Then the Marsden-Weinstein theorem [10, 11] ensures that the symplectic reduction of P by the group G, denoted by the double quotient $P /\!/ G$, is still a symplectic manifold and carries a unique symplectic form. The reduced phase space is given by imposing the constraints and dividing the constraint surface by the action of gauge transformations. This is written as

$$P /\!/ G \equiv H^{-1}(0)/G. \tag{1.6}$$

For notational simplicity, we will denote the group of transformations G and the associated Hamiltonian H with the same letters.

In the case of four-dimensional gravity, the physical phase space is obtained from the kinematical (unconstrained) phase space $\mathcal{P}$ by performing three symplectic reductions. The first one is defined with respect to the group of SU(2) gauge transformations $\mathcal{G}\equiv C^\infty\big(\Sigma,\mathrm{SU}(2)\big)$. Since the action of this gauge group on $\mathcal{P}$ is Hamiltonian, we can define the gauge-invariant phase space $T^*\mathcal{A} /\!/ \mathcal{G}$. More precisely, the Hamiltonian generating these transformations is the smeared Gauss constraint:

$$\mathcal{G}(\alpha)=\int_\Sigma \alpha^i(\mathrm{d}_A E)_i=0, \tag{1.7}$$

where d_A denotes the covariant differential and α is a Lie algebra-valued function. Its infinitesimal action on the phase space variables is given by

$$\delta^{\mathcal{G}}_\alpha A=\big\{A,\mathcal{G}(\alpha)\big\}=\mathrm{d}_A\alpha,\qquad\qquad \delta^{\mathcal{G}}_\alpha E=\big\{E,\mathcal{G}(\alpha)\big\}=[E,\alpha]. \tag{1.8}$$

The other relevant symplectic reduction is defined with respect to the group of spatial diffeomorphisms, and enables one to construct the diffeomorphism-invariant phase space $T^*\mathcal{A} /\!/ \big(\mathcal{G}\times\mathrm{Diff}(\Sigma)\big)$. Here, the action of the group of diffeormorphisms on the phase space variables is given by

$$\delta^{\mathcal{D}}_\xi A=\big\{A,\mathcal{D}(\xi)\big\}=\mathcal{L}_\xi A,\qquad\qquad \delta^{\mathcal{D}}_\xi E=\big\{E,\mathcal{D}(\xi)\big\}=\mathcal{L}_\xi E, \tag{1.9}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ^a. This group is generated through Poisson brackets with the Hamiltonian

$$\mathcal{D}(\xi)=\mathcal{H}(\xi)-\mathcal{G}(\xi^a A_a),\qquad\qquad \text{with}\qquad \mathcal{H}(\xi)=\int_\Sigma \xi^a F^i_{ab}E^b_i. \tag{1.10}$$

Finally, the physical phase space can be obtained from the gauge and diffeomorphism-invariant phase space by performing a symplectic reduction with respect to the scalar constraint. This latter is given by

$$\mathcal{H}(N)=\int_\Sigma N\frac{E^a_iE^b_j}{2\sqrt{\det(E^a_i)}}\left(\epsilon^{ij}{}_kF^k_{ab}-2(\gamma^2-\sigma)K^i_{[a}K^j_{b]}\right)=0, \tag{1.11}$$

where the smearing variable is the lapse function N, and $\sigma=\mp1$ in Lorentzian or Riemannian signature respectively. Notice that for a (anti) self-dual connection ($\gamma=\pm i$ in the Lorentzian case,

or ± 1 in the Riemannian case) the second term vanishes and the constraint simplifies greatly.

II. SPIN NETWORK PHASE SPACE

In loop gravity, one does not work directly with the continuous kinematical Hilbert space, but instead with the projective limit of Hilbert spaces associated to embedded oriented graphs Γ [4, 12]. The Hilbert space associated with one graph is the so-called spin network Hilbert space. It represents a truncation of the full Hilbert space to a finite number of degrees of freedom. What we would like to emphasize here is that spin network Hilbert spaces can be obtained as the quantization of finite-dimensional phase spaces associated to embedded oriented graphs Γ. Each of these truncated phase spaces form the so-called holonomy-flux algebra. This fact has already been recognized in the literature [9] and is at the basis of most of the recent semi-classical analyses of LQG [13–15]. Our main point is that the process of truncating the theory to a finite number of degrees of freedom and the process of quantizing this truncated theory are separate constructions which have to be studied individually. Here we would like to adopt the point of view that the continuous kinematical phase space $\mathcal{P}$ can be described as the projective limit of phase spaces P_Γ associated to embedded oriented graphs Γ. In particular, we would like to understand the relationship between these finite-dimensional phase spaces P_Γ and the continuous phase space variables.

An oriented graph Γ is defined as a one-cellular complex [16] consisting of a set E_Γ of oriented edges e (one-dimensional submanifolds of Σ) and a set V_Γ of vertices v. The end points of the oriented edges are the vertices, and we denote by s, t the two functions assigning a source vertex $s(e)$ and a target vertex $t(e)$ to each edge e. We also denote by e^{-1} the edge e with a reverse orientation. The kinematical spin network phase space P_Γ associated with such a graph is isomorphic to a direct product for each edge of SU(2) cotangent bundles[1]:

$$P_\Gamma \equiv \mathop{\times}_e T^*\mathrm{SU}(2)_e. \tag{2.1}$$

Explicitly, this phase space is labeled by couples $(h_e, X_e) \in \mathrm{SU}(2) \times \mathfrak{su}(2)$ of Lie group and Lie algebra elements for each edge $e \in \Gamma$. This data depends on a choice of orientation for each edge, and under an orientation reversal we have

$$h_{e^{-1}} = h_e^{-1}, \qquad\qquad X_{e^{-1}} = -h_e^{-1} X_e h_e. \tag{2.2}$$

Since we have chosen here to trivialize $T^*\mathrm{SU}(2)$ with right-invariant vector fields, this last relation means that under orientation reversal of the edge we obtain the left-invariant ones. The variables (h_e, X_e) satisfy the Poisson algebra

$$\left\{X_e^i, X_{e'}^j\right\} = \delta_{ee'}\epsilon^{ij}{}_k X_e^k, \qquad \left\{X_e^i, h_{e'}\right\} = -\delta_{ee'}\tau^i h_e + \delta_{ee'^{-1}} h_e \tau^i, \qquad \left\{h_e, h_{e'}\right\} = 0, \tag{2.3}$$

where we have used notations such that[2] $X_e \equiv X_e^i \tau_i$. As shown in [18, 19], the symplectic potential and symplectic two-form for this Poisson structure are given respectively by

$$\Theta_\Gamma = \sum_e \mathrm{Tr}\left(X_e \mathrm{d}h_e h_e^{-1}\right), \qquad\qquad \Omega_\Gamma = -\mathrm{d}\Theta_\Gamma. \tag{2.4}$$

[1] Given a Lie group G, the group action on itself by left (or right) multiplication can be used to obtain an isomorphism of vector fields with the Lie algebra $\mathfrak{g}$, and to trivialize the cotangent bundle as $T^*G = G \times \mathfrak{g}^*$ [17].

[2] In this work, we define $\tau_i = -i\sigma_i/2$, where σ_i are the Pauli matrices. The $\mathfrak{su}(2)$ commutation relations are then given by $[\tau_i, \tau_j] = \epsilon_{ij}{}^k \tau_k$, where $\epsilon_{ij}{}^k$ is the completely antisymmetric Levi-Civita tensor.

On the spin network phase space P_Γ, we can define the action of the gauge group $G_\Gamma \equiv \mathrm{SU}(2)^{|V_\Gamma|}$ at the vertices V_Γ of the graph. Given an element $g_v \in \mathrm{SU}(2)$, finite gauge transformations are given by

$$g_v \triangleright h_e = g_{s(e)} h_e g_{t(e)}^{-1}, \qquad\qquad g_v \triangleright X_e = g_{s(e)} X_e g_{s(e)}^{-1}, \tag{2.5}$$

where $s(e)$ (resp. $t(e)$) denotes the starting (resp. terminal) vertex of e. This action on the variables h_e and X_e is generated at each vertex by the Hamiltonian

$$G_v \equiv \sum_{e \ni v} X_e = \sum_{e|s(e)=v} X_e + \sum_{e|t(e)=v} X_{e^{-1}}, \tag{2.6}$$

which can be understood as a discrete Gauss constraint. Since this action is Hamiltonian, we can define the gauge-invariant phase space

$$P_\Gamma^G = \underset{e}{\times} T^*\mathrm{SU}(2)_e \,/\!\!/\, \mathrm{SU}(2)^{|V_\Gamma|} = G_v^{-1}(0)/\mathrm{SU}(2)^{|V_\Gamma|} \tag{2.7}$$

by symplectic reduction where, as explained above, the double quotient means imposing the Gauss constraint at each vertex v and then dividing out the action of the SU(2) gauge transformation (2.5) that it generates.

The question we would like to address is: What is the relationship between the continuous phase space $\mathcal{P}$ described in the previous section, and the spin network phase space P_Γ? More precisely, we would like to know if it is possible to reconstruct from the discrete data P_Γ a point in the continuous phase space $\mathcal{P}$? In order to describe the relationship between the discrete and continuous data, we need a map from the continuous to the discrete phase space. We can then study its kernel and see to what extent it can be inverted. This is the object of the next sections.

A. From continuous to discrete data

In order to construct the discrete data, let us first choose an embedding $f_\Gamma : \Gamma \longrightarrow \Sigma$ of the graph Γ into the spatial manifold Σ. Given this embedding, it is well understood in the discrete picture that the group elements h_e represent holonomies of the Ashtekar-Barbero connection A_a^i along edges e. It is necessary to work with such objects because an important step toward the quantization of the canonical theory is the smearing of the Poisson algebra (1.3). Since the connection A_a^i is a one-form, it is natural to smear it along paths e. Now we could just take the integral of A along e as a smearing but this will not respect the gauge transformations. What is needed is a smearing that does intertwine the notion of continuous and discrete gauge transformations. It is well known that this is given by the notion of parallel transport along e, encoded in the holonomy

$$h_e(A) \equiv \overrightarrow{\exp} \int_e A = \overrightarrow{\exp} \int_e A_a^i \dot{e}^a \tau_i = \overrightarrow{\exp} \int_{s(e)}^{t(e)} A_a^i \dot{e}^a \tau_i, \tag{2.8}$$

where $\dot{e}^a$ denotes the tangent vector to the path and $\overrightarrow{\exp}$ denotes the path-ordered exponential.

Let us recall some fundamental properties of the holonomy functional. The holonomy is invariant under reparametrizations of the path e, and the holonomy of a path corresponding to a single point is the identity. If we consider the composition $e = e_1 \circ e_2$ of two paths which are such that $s(e_2) = t(e_1)$, the holonomy satisfies

$$h_e = h_{e_1} h_{e_2}. \tag{2.9}$$

If we reverse the orientation of a path, we have

$$h_{e^{-1}} = h_e^{-1}. \tag{2.10}$$

These properties come from the fact that the holonomy is a representation of the groupoid of oriented paths [20]. Under SU(2) gauge transformations, the holonomy transforms as

$$g \triangleright h_e = g_{s(e)} h_e g_{t(e)}^{-1}, \tag{2.11}$$

which shows that the finite gauge transformation $g \triangleright A = gAg^{-1} + g\mathrm{d}g^{-1}$ of the connection becomes a discrete gauge symmetry acting on the vertices defining the boundary of the edge e. Finally, under the action of a diffeomorphism $\Phi \in \mathrm{Diff}(\Sigma)$, the holonomy transforms as

$$h_e(\Phi^* A) = h_{\Phi(e)}(A). \tag{2.12}$$

The exact meaning of "momentum" variable X_e is less clear. Roughly speaking, we usually build a flux operator by smearing the field E_i^a along a surface F_e dual to an edge e [2]. But if one wants this integrated flux to have a covariant behavior under gauge transformations, it is essential for the integration along F_e to involve some notion of parallel transport. Indeed, the naive definition

$$X_e^i(E) = \int_{F_e} E^i(x) \tag{2.13}$$

of the flux is not covariant under gauge transformations, i.e.

$$X_e^i(g \triangleright E) = X_e^i\left(gEg^{-1}\right) \neq g_{s(e)} X_e^i(E) g_{s(e)}^{-1}. \tag{2.14}$$

This is an important point which has often been ignored in the LQG literature, the only noticeable exceptions being [21], and more recently [5, 22]. For the holonomy, the only reason we consider the paralell transport operator instead of the simple integral of A along e is to have a discretisation covariant under gauge transformation. It is as important to preserve this covariance for the flux than it is for the holonomy.

The way around this problem is to define a flux operator which also depends on the connection through its holonomy. Given an oriented edge $e \in \Gamma$ and a point u on this edge, we choose a surface F_e intersecting e transversally at $u = F_e \cap e$. We also choose a set of paths π_e assigning to any point $x \in F_e$ a unique path π_e going from the source $s(e)$ to x. Such a path starts at the source vertex of the edge e, goes along e until it reaches the intersection point $u = F_e \cap e$, and then goes from u to any point $x \in F_e$ while staying tangential to the surface F_e. More precisely, we have $\pi_e : F_e \times [0,1] \longrightarrow \Sigma$ such that $\pi_e(x,0) = s(e)$ and $\pi_e(x,1) = x$. With the set of data (F_e, π_e), one can define the flux operator

$$X^i_{(F_e,\pi_e)}(A,E) \equiv \int_{F_e} h_{\pi_e}(x) E^i(x) h_{\pi_e}^{-1}(x), \tag{2.15}$$

where

$$h_{\pi_e}(x) \equiv \overrightarrow{\exp} \int_{s(e)}^{x} A. \tag{2.16}$$

Notice that by definition, the source of the path π_e is $s(e)$, and its target is the point $x \in F_e$.

Therefore, under the gauge transformations

$$g \triangleright E(x) = g(x)E(x)g(x)^{-1}, \qquad\qquad g \triangleright h_{\pi_e}(x) = g_{s(e)}h_{\pi_e}(x)g(x)^{-1}, \tag{2.17}$$

the flux operator becomes

$$X_{(F_e,\pi_e)}(g \triangleright A, g \triangleright E) = g_{s(e)}X_{(F_e,\pi_e)}(A,E)g^{-1}_{s(e)}, \tag{2.18}$$

which is in agreement with (2.5). The existence of a covariant transformation property is one of the main justifications for introducing the extra holonomy dependance in the definition of the flux operator. With the definition (2.15), the flux operator intertwines the continuous and discrete actions of the gauge group.

One can see from the definition of the paths π_e that under an orientation reversal of the edge we have $\pi_{e^{-1}} = e^{-1} \circ \pi_e$, and therefore

$$h_{\pi_{e^{-1}}}(x) = h_e^{-1}h_{\pi_e}(x). \tag{2.19}$$

Moreover, the surface F_e possesses a reverse orientation $F_{e^{-1}} = -F_e$, and thus we have

$$X_{\left(F_{e^{-1}},\pi_{e^{-1}}\right)} = -h_e^{-1}X_{(F_e,\pi_e)}h_e, \tag{2.20}$$

which proves that our mapping is consistent with (2.2). Notice also that any two fluxes which differ only by the choice of paths and surfaces, but possess the same intersection with e, are in the commutant of the holonomy algebra:

$$\left\{X_{(F'_e,\pi'_e)} - X_{(F_e,\pi_e)}, h_e\right\} = 0, \qquad F_e \cap e = F'_e \cap e \tag{2.21}$$

Finally, one can see that the mapping we have described reproduces the Poisson algebra (2.3). Indeed, if we choose the surface F_e such that the intersecting vertex $u = F_e \cap e$ approaches $s(e)$, we obtain

$$\left\{X^i_{(F_e,\pi_e)}(A,E), h_{e'}(A)\right\} = -\delta_{ee'}\tau^i h_e(A). \tag{2.22}$$

Finally, we know that the requirement of consistency with the Jacobi identity imposes that the fluxes do not commute among each other. This property, which seems inconsistent if one thinks of X_e as depending purely on the (commuting) densitized triad field, is perfectly understandable once we realize that the flux depends also on the connection, and explains the "mystery" behind the non-commutativity of the fluxes [23]. This is consistent with the understanding of the spin network phase space in terms of twisted geometries [5], where it appears clearly that the flux operators also contain information about the holonomies, and cannot be thought of as being purely geometrical. In other words, the flux operators are not commuting because they capture information not only about the intrinsic geometry, but also about the extrinsic curvature.

The map that we have described depends on three types of data. It depends on a choice of embedding f_Γ of Γ into Σ, a choice of surface F_e transverse to the edge e at u, and a choice of path π_e going from $s(e)$ to a point $x \in F_e$. Once this data is given, we can construct a map

$$\begin{array}{rccl} \mathcal{I}: & \mathcal{P} & \longrightarrow & P_\Gamma \\ & (A,E) & \longmapsto & \big(h_e(A), X_{(F_e,\pi_e)}(A,E)\big)\,, \end{array} \tag{2.23}$$

which has the key property of intertwining gauge transformations on the continuous and discrete phase spaces, is compatible with the orientation reversal of the edges, and respects the Poisson

structure of $T^*\mathrm{SU}(2)$.

B. From discrete to continuous data

Now we would like to investigate to what extent it is possible to invert the map from continuous to discrete data $\mathcal{I} : \mathcal{P} \longrightarrow P_\Gamma$. In other words, to what extent does the discrete data determine the continuous data? Can we reconstruct a unique representative of the continuous data starting from the discrete one, or describe a specific equivalence class?

At first sight, this seems like an impossible task. Indeed, if one first focuses on the connection, one needs to choose an embedding f_Γ to construct the holonomies, so there is no way the discrete group elements will determine the connection unless we know this embedding. Moreover, one clearly sees that the flux operator is not uniquely defined by the electric field E. There are several ambiguities in its definition. There are many possible choices of surfaces F_e that are transverse to the edge e, and also many possible paths that one can choose on F_e. Different choices lead to different mappings from the continuous data to the discrete data. This means that giving a flux $X_{(F_e,\pi_e)}$ (which we will call X_e for simplicity) does not allow one to reconstruct a continuous field E, which constitutes a fundamental ambiguity. This state of affairs is fine if one treats the discrete data as some approximate description of continuous geometry which only takes physical meaning in some continuous limit. This is the usual point of view [9], and it implies that operators expressed in terms of the fluxes X_e do not have a sharp semi-classical geometric interpretation.

In this work we would like to be more ambitious and interpret the discrete data as potential initial value data for the continuous theory of gravity. The challenge is to show that one can reconstruct continuous fields (A, E) explicitly from the knowledge of the discrete data (h_e, X_e). How can this be possible in light of all the ambiguities that we have listed above? In order to make some progress in this direction, let us first remark that there are configurations of fields for which the ambiguities disappear. This is the case in particular for a flat connection.

Suppose that we focus on a region C_v of simple topology (isomorphic to a three-ball) around a vertex $v \in C_v$, and that in this region the connection A is flat. In this case, the expression (2.15) for the flux becomes independent of the system of paths π_e, since the flatness of the connection implies that there exists an SU(2) element $a(x)$ such that $A = a\mathrm{d}a^{-1}$ and $h_{\pi_e}(x) = a(v)a(x)^{-1}$. Indeed, we have

$$X^i_{(F_e,\pi_e)} = X^i_{F_e} = a(v)\left(\int_{F_e} a(x)^{-1}E^i(x)a(x)\right)a(v)^{-1}, \tag{2.24}$$

and the dependence on the system of paths has disappeared. Moreover, one can see that the Gauss law expresses the fact that $X^i_{F_e} = X^i_{F'_e}$, for if F_e and F'_e have the same oriented boundary, their union encloses a volume ∂C_v and we have that:

$$0 = \int_{C_v} a(x)^{-1}\mathrm{d}_A E^i(x)a(x) = \int_{C_v} \mathrm{d}\left(a(x)^{-1}E^i(x)a(x)\right) = a(v)^{-1}\left(X^i_{F_e} - X^i_{F'_e}\right)a(v). \tag{2.25}$$

In the next section, we are going to make this statement more precise, and study the case of a partially flat connection.

III. PARTIALLY FLAT CONNECTION

In this section, we formulate and prove the equivalence between the continuous phase space of partially flat geometries and the discrete spin network phase space. In order to do so, we first need

to introduce some notions of topology.

Definition 1. *A cellular decomposition Δ of a space Σ is a decomposition of Σ as a disjoint union (partition) of open-cells of varying dimension satisfying the following conditions:*

i) An n-dimensional open cell is a topological space which is homeomorphic to the n-dimensional open ball.

ii) The boundary of the closure of an n-dimensional cell is contained in a finite union of cells of lower dimension.

The n-skeleton Δ_n of a cellular decomposition is the union of cells of dimension less than or equal to n.

Clearly, the n-skeleton of a cellular decomposition is also a cellular decomposition. In particular, the one-skeleton Δ_1 of a cellular decomposition is a graph. Let us now suppose that we have a graph Γ embedded in Σ. We need to introduce the notion of a cellular decomposition dual to Γ.

Definition 2. *A cellular decomposition Δ of a three-dimensional space Σ is said to be dual to the graph Γ if there is a one-to-one correspondence $v \longmapsto C_v$ between vertices of Γ and three-cells of Δ, and a one-to-one correspondence $e \longmapsto F_e$ between edges of Γ and two-cells of Δ, such that:*

i) There is a unique vertex v inside each three-cell C_v.

ii) The two-cells F_e intersect Γ transversally at one point only, and the intersection belongs to the interior of the edge e of Γ.

In other words, a cellular decomposition dual to Γ is such that each vertex of Γ is dual to a three-cell, and each edge of Γ is dual to a two-cell. Finally, let us consider a pair (Γ,Γ^*) of graphs embedded in Σ.

Definition 3. *We say that an embedded graph Γ^* is dual to the embedded graph Γ, or that (Γ,Γ^*) forms a pair of dual graphs, if there exists a cellular decomposition Δ dual to Γ, whose one-skeleton Δ_1 is Γ^*.*

From now on, we consider that (Γ,Γ^*) is a pair of dual embedded graphs, and we denote by Δ the cellular decomposition dual to Γ with a one-skeleton Δ_1 given by Γ^*. Notice that if we take any diffeomorphism Φ_o on Σ which does not act on Γ^* or the vertices of Γ, we obtain an equivalent[3] cellular decomposition $\Phi_o(\Delta)$. Given such a pair of dual graphs, we are going to construct a certain phase space $\mathcal{P}_{\Gamma,\Gamma^*}$, and prove that it is the continuous analogue of the discrete spin network phase space P_Γ. In fact, we are going to show that there is a symplectomorphism between $\mathcal{P}_{\Gamma,\Gamma^*}$ and P_Γ.

A. The reduced phase space $\mathcal{P}_{\Gamma,\Gamma^*}$

To define the reduced phase space $\mathcal{P}_{\Gamma,\Gamma^*}$, we first construct a group $\mathcal{F}_{\Gamma^*}\times\mathcal{G}_\Gamma$ of gauge transformations acting on $\mathcal{P}$. For this, let us consider an infinite-dimensional Abelian group of transformations $\mathcal{F}_{\Gamma^*}$ parametrized by Lie algebra-valued one-forms $\phi^i\in\Omega^1\big(\Sigma,\mathfrak{su}(2)\big)$ which have the property that they vanish on Γ^*:

$$\phi^i(x)=0, \qquad \forall\, x\in\Gamma^*. \tag{3.1}$$

This group action is Hamiltonian and generated by the curvature constraint

$$\mathcal{F}_{\Gamma^*}(\phi)=\int_\Sigma \phi_i\wedge F^i(A), \tag{3.2}$$

[3] Since these diffeomorphisms vanish on Γ^*, the duality between edges and faces is preserved.

whose action on the continuous phase space $\mathcal{P}$ is given by

$$\delta_\phi^{\mathcal{F}_{\Gamma^*}} A = \big\{A, \mathcal{F}_{\Gamma^*}(\phi)\big\} = 0, \qquad\qquad \delta_\phi^{\mathcal{F}_{\Gamma^*}} E = \big\{E, \mathcal{F}_{\Gamma^*}(\phi)\big\} = \mathrm{d}_A\phi. \tag{3.3}$$

This constraint enforces the flatness of the connection outside of the one-skeleton graph Γ^*. The second group, $\mathcal{G}_\Gamma$, is the group of gauge transformations parametrized by Lie algebra-valued functions $\alpha^i \in \Omega^0\big(\Sigma, \mathfrak{su}(2)\big)$ which have the property that they vanish on the vertices of Γ:

$$\alpha^i(x) = 0, \qquad\qquad \forall\, x \in V_\Gamma. \tag{3.4}$$

This group action is also Hamiltonian. It is generated by the smeared Gauss constraint

$$\mathcal{G}_\Gamma(\alpha) = \int_\Sigma \alpha^i(\mathrm{d}_A E)_i, \tag{3.5}$$

whose infinitesimal action on the phase space variables is given by

$$\delta_\alpha^{\mathcal{G}_\Gamma} A = \big\{A, \mathcal{G}_\Gamma(\alpha)\big\} = \mathrm{d}_A\alpha, \qquad\qquad \delta_\alpha^{\mathcal{G}_\Gamma} E = \big\{E, \mathcal{G}_\Gamma(\alpha)\big\} = [E, \alpha]. \tag{3.6}$$

From the various Poisson brackets

$$\big\{\mathcal{G}_\Gamma(\alpha), \mathcal{G}_\Gamma(\alpha')\big\} = \mathcal{G}_\Gamma([\alpha, \alpha']), \tag{3.7a}$$
$$\big\{\mathcal{G}_\Gamma(\alpha), \mathcal{F}_{\Gamma^*}(\phi)\big\} = \mathcal{F}_{\Gamma^*}([\alpha, \phi]), \tag{3.7b}$$
$$\big\{\mathcal{F}_{\Gamma^*}(\phi), \mathcal{F}_{\Gamma^*}(\phi')\big\} = 0, \tag{3.7c}$$

we see that the Hamiltonians (3.2) and (3.5) form a first class algebra.

We are interested in the phase space obtained from $\mathcal{P}$ by symplectic reduction with respect to $\mathcal{F}_{\Gamma^*}$ and $\mathcal{G}_\Gamma$, which we denote by

$$\mathcal{P}_{\Gamma,\Gamma^*} \equiv T^*\mathcal{A} /\!\!/ \big(\mathcal{F}_{\Gamma^*} \times \mathcal{G}_\Gamma\big) = \mathcal{C}_{\Gamma,\Gamma^*}/\big(\mathcal{F}_{\Gamma^*} \times \mathcal{G}_\Gamma\big), \tag{3.8}$$

where

$$\mathcal{C}_{\Gamma,\Gamma^*} \equiv \big\{(A, E) \in T^*\mathcal{A} | F(A)(x) = \mathrm{d}_A E(y) = 0,\ \forall\, x \in \Sigma\backslash\Gamma^*,\ \forall\, y \in \Sigma\backslash V_\Gamma\big\} \tag{3.9}$$

is the constrained space. This is the infinite-dimensional space of flat SU(2) connections on $\tilde{\Sigma} \equiv \Sigma\backslash\Gamma^*$, and fluxes satisfying the Gauss law outside of V_Γ. Once we divide this constrained space by the action of the two gauge groups introduced above, we obtain the finite-dimensional orbit space $\mathcal{P}_{\Gamma,\Gamma^*}$. We are going to prove that $\mathcal{P}_{\Gamma,\Gamma^*}$ is the continuous analogue of the discrete spin network phase space P_Γ.

Let us start by constructing a three-dimensional cellular decomposition of the region. Since we have chosen Γ^* to be the one-skeleton Δ_1 of the cellular decomposition Δ of Σ, the cellular decomposition of $\tilde{\Sigma}$ is simply given by $\tilde{\Delta} \equiv \Delta\backslash\Delta_1$. Explicitly, the decomposition $\tilde{\Delta}$ can be written as

$$\tilde{\Delta} = \bigcup_v C_v \bigcup_e F_e, \tag{3.10}$$

where C_v are three-dimensional open cells labeled by the vertices $v \in \Gamma$, and F_e are two-dimensional open cells labeled by the edges $e \in \Gamma$. We would like to solve the curvature constraint $F(A) = \mathrm{d}_A A = 0$ on $\tilde{\Sigma}$ and the Gauss constraint $\mathrm{d}_A E = 0$ on $\Sigma\backslash V_\Gamma$. We start by solving them for each three-dimensional cell C_v.

To solve the curvature constraint, let us define on a three-cell C_v a group-valued map $a_v(x): C_v \longrightarrow \mathrm{SU}(2)$ as the path-ordered exponential

$$a_v(x) \equiv \overrightarrow{\exp} \int_x^v A, \tag{3.11}$$

where the integration can be taken over any arbitrary path from the point $x \in C_v$ to the vertex v because the connection is flat and C_v is simply connected. By construction, this map is such that $a_v(v) = 1$. This allows us to reconstruct on C_v the flat connection A as

$$A(x) = a_v(x)\mathrm{d}a_v^{-1}(x). \tag{3.12}$$

The second constraint to satisfy is the Gauss law outside of the vertex v which lies inside the cell C_v. Because the connection is flat, the covariant derivative of the electric field E can be written as

$$\mathrm{d}_A E = dE + \left[a_v \mathrm{d}a_v^{-1}, E\right] = a_v \mathrm{d}\left(a_v^{-1} E a_v\right) a_v^{-1} = a_v \mathrm{d}X_v a_v^{-1}, \tag{3.13}$$

where we have introduced the Lie algebra-valued two-form field

$$X_v(x) \equiv a_v(x)^{-1} E(x) a_v(x). \tag{3.14}$$

Therefore, we see that the Gauss law implies that the two-form X_v is closed outside of v since

$$\mathrm{d}X_v(x) = a_v(x)^{-1}\mathrm{d}_A E(x) a_v(x) = 0, \qquad\qquad \forall\, x \in C_v - \{v\}. \tag{3.15}$$

The electric field can now easily be reconstructed since we have

$$E(x) = a_v(x) X_v(x) a_v(x)^{-1}. \tag{3.16}$$

One can conclude that a general solution of the two constraints $F(A) = \mathrm{d}_A A = 0$ and $\mathrm{d}_A E = 0$ on C_v and $C_v - \{v\}$ respectively, is given in terms of a Lie algebra-valued closed two-form X_v and a group element $a_v : C_v \longrightarrow \mathrm{SU}(2)$, the connection and flux fields being given by (3.12) and (3.16).

Now we can extend this solution to the whole space $\tilde{\Sigma}$ by gluing consistently the solutions on each cell. We have labeled the three-dimensional cells C_v with vertices of the graph Γ. Consequently, the two-dimensional cells F_e, labeled by edges $e = (v_1 v_2)$ of Γ connecting two vertices (such that $s(e) = v_1$ and $t(e) = v_2$), are obtained by intersecting two three-dimensional cells as

$$F_e = \overline{C_{v_1}} \cap \overline{C_{v_2}}, \tag{3.17}$$

where the bar denotes the closure of the cell. We assume that the two-dimensional cells F_e are oriented, and that their orientation is reversed when we change the orientation of the edge e. Demanding that the connection and flux fields be continuous across the two-dimensional cells amounts to assuming that there exists, for each F_e, an SU(2) element h_e such that

$$a_{v_2}(x) = a_{v_1}(x) h_e, \qquad\qquad X_{v_2}(x) = h_e^{-1} X_{v_1}(x) h_e, \tag{3.18}$$

for $x \in F_e$. Notice that the first equality can be written as

$$h_e(A) = a_{s(e)}(x)^{-1} a_{t(e)}(x) = \overrightarrow{\exp} \int_e A, \tag{3.19}$$

where x is any point on the two-cell F_e, and once again the definition does not depend on x because the connection is flat. By construction, one can see that under an orientation reversal we have

$h_{e^{-1}} = h_e^{-1}$.

This construction shows that the constrained space $\mathcal{C}$ is isomorphic to the data (a_v, X_v, h_e), subject to the conditions (3.18). We are now interested in the quotient of this constraint space by the gauge group $\mathcal{F}_{\Gamma^*} \times \mathcal{G}_\Gamma$. Elements of this gauge group are pairs $\big(\phi(x), g_o(x)\big)$, where ϕ is a Lie algebra-valued one-form which vanishes on Γ^*, and g_o is an element of SU(2) (obtained by exponentiation of α) fixed to the identity of the group at the vertices V_Γ. The action of $\mathcal{F}_{\Gamma^*} \times \mathcal{G}_\Gamma$ on the pair $(A, E) \in \mathcal{P}$ translates on the constraint surface $\mathcal{C}$ into an action on the data (a_v, X_v, h_e) given by

$$a_v(x) \longrightarrow g_o(x) a_v(x), \qquad X_v(x) \longrightarrow X_v(x) + \mathrm{d}\left(a(x)^{-1}\phi(x)a(x)\right), \qquad h_e \longrightarrow h_e. \tag{3.20}$$

Following (2.15), let us compute the flux X_e across a surface dual to an edge e which is such that $s(e) = v$. It is given by

$$X_e = \int_{F_e} h_{\pi_e}(x)E(x)h_{\pi_e}(x)^{-1} = \int_{F_e} a_v(v)a_v(x)^{-1}E(x)a_v(x)a_v(v)^{-1} = \int_{F_e} X_v, \tag{3.21}$$

where we have used the fact that $a_v(v) = 1$. We see that the observables which are invariant under this gauge transformation are simply given by the holonomies h_e and the fluxes X_e.

B. The symplectomorphism between $\mathcal{P}_{\Gamma,\Gamma^*}$ and P_Γ

Now we come to our main result, which is the symplectomorphism between the continuous phase space $\mathcal{P}_{\Gamma,\Gamma^*}$ and the discrete spin network phase space P_Γ. Let us construct a map between the constrained continuous data in $\mathcal{C}_{\Gamma,\Gamma^*}$ (see (3.9)) and discrete data on the spin network phase space P_Γ, and denote it by

$$\begin{array}{rrcl} \mathcal{I}: & \mathcal{C}_{\Gamma,\Gamma^*} & \longrightarrow & P_\Gamma \\ & (A, E) & \longmapsto & \big(h_e(A), X_e(A, E)\big). \end{array} \tag{3.22}$$

For this, we define for every three-cell C_v a group-valued map $a_v : C_v \longrightarrow \mathrm{SU}(2)$ such that $a_v(v) = 1$ and a Lie algebra-valued two-form $X_v : C_v \longrightarrow \Omega^2\big(C_v, \mathfrak{su}(2)\big)$ closed outside te vertices of Γ. Given these fields, we can reconstruct on C_v the connection and the two-form field using

$$A(x) = a_v(x)\mathrm{d}a_v(x)^{-1}, \qquad E(x) = a_v(x)X_v(x)a_v(x)^{-1}. \tag{3.23}$$

The map $\mathcal{I}$ is then defined by

$$h_e(A) \equiv \overrightarrow{\exp}\int_e A = a_{s(e)}(x)^{-1}a_{t(e)}(x), \tag{3.24a}$$

$$X_e(A, E) \equiv \int_{F_e} h_{\pi_e}(x)E(x)h_{\pi_e}(x)^{-1} = \int_{F_e} X_{s(e)}(x), \tag{3.24b}$$

where in the definition of h_e, x is any point on the two-cell F_e, and once again the definition does not depend on x because the connection is flat. To compute the holonomy h_e, we have used the group elements $a_{s(e)}(x)$ and $a_{t(e)}(x)$ to define the connection on the two cells dual to the vertices $s(e)$ and $t(e)$ respectively.

It is possible to use equation (3.24b) to write down the relationship between the discrete and continuous Gauss laws. We already know from (3.15) that the Gauss law is equivalent to the

requirement that the two-form X_v be closed outside of the vertex v. We can now write that

$$\int_{C_v} a_v(x)^{-1}\mathrm{d}_A E(x) a_v(x) = \int_{C_v} \mathrm{d}X_v = \int_{\cup_e F_e = \partial C_v} X_{s(e)} = \sum_{e|s(e)=v} X_e = G_v, \tag{3.25}$$

which relates the continuous and discrete constraints. This shows that the violation of the continuous Gauss constraint is located at the vertices of Γ, and given by a distribution determined by the discrete Gauss constraint:

$$\mathrm{d}_A E(x) = \sum_{v\in V_\Gamma} G_v \delta(x-v). \tag{3.26}$$

Since the map $\mathcal{I}$ is invariant under the gauge transformations $\mathcal{F}_{\Gamma^*}\times\mathcal{G}_\Gamma$ we can write it as a map

$$[\mathcal{I}]:\mathcal{P}_{\Gamma,\Gamma^*}\longrightarrow P_\Gamma.$$

We will now show that this map is not only invertible, but also a symplectomorphism.

Proposition 1. *The map $[\mathcal{I}]:\mathcal{P}_{\Gamma,\Gamma^*}\longrightarrow P_\Gamma$ defined by (3.24) is a symplectomorphism, and is invariant under the action of diffeomorphisms connected to the identity preserving Γ^* and the set V_Γ of vertices of Γ.*

We are going to prove this proposition in the remainder of this work. Before doing so, let us stress that this result implies the existence of an inverse map which allows one to reconstruct from the discrete data an equivalence class $[A(h_e),E(h_e,X_e)]$ of continuous configurations satisfying the curvature and Gauss constraints (i.e. configurations in the constrained space $\mathcal{C}$). Explicitly, this equivalence class is defined with respect to the equivalence relation

$$(A,E)\sim\big(g\triangleright A, g_o^{-1}(E+\mathrm{d}_A\phi)g_o\big), \tag{3.27}$$

where once again ϕ is a Lie algebra-valued one-form vanishing on Γ^*, and g_o is an element of SU(2) fixed to the identity of the group at the vertices V_Γ. By construction, we see that the map $\mathcal{I}$ intertwines the notion of gauge transformations, i.e. satisfies $\mathcal{I}(g\triangleright A, g\triangleright E)=g\triangleright\mathcal{I}(A,E)$.

Evidently, proposition (1) implies a similar proposition for the gauge-invariant phase spaces. Indeed, if one defines

$$\mathcal{P}^{\mathcal{G}}_{\Gamma,\Gamma^*}\equiv T^*\mathcal{A}\,/\!/\,\big(\mathcal{F}_{\Gamma^*}\times\mathcal{G}\big)=\mathcal{C}^{\mathcal{G}}_{\Gamma^*}/\big(\mathcal{F}_{\Gamma^*}\times\mathcal{G}\big), \tag{3.28}$$

where

$$\mathcal{C}^{\mathcal{G}}_{\Gamma^*}\equiv\big\{(A,E)\in T^*\mathcal{A}|F(A)(x)=\mathrm{d}_A E(y)=0,\ \forall\, x\in\Sigma\backslash\Gamma^*,\ \forall\, y\in\Sigma\big\}, \tag{3.29}$$

and $\mathcal{G}=C^\infty\big(\Sigma,\mathrm{SU}(2)\big)$ is the group of full SU(2) gauge transformations, we have the symplectomorphism $\mathcal{P}^{\mathcal{G}}_{\Gamma,\Gamma^*}=P^G_\Gamma$ between the continuous and discrete gauge-invariant phase spaces. This follows directly from proposition (1), and the fact that $\mathcal{G}=\mathcal{G}_\Gamma\times G_\Gamma$, where G_Γ is the group of discrete gauge transformations acting at the vertices $v\in V_\Gamma$ only.

Notice that when we act with the full group $\mathcal{G}$ of SU(2) transformations, the holonomies h_e and the fluxes X_e clearly become gauge-covariant. Indeed, since the group element g is not fixed to the identity at the vertices v anymore, we have $g\triangleright a_v(x)=g(x)a_v(x)g(v)^{-1}$, and therefore the definition (3.18) tells us that we have $g\triangleright h_e=g_{v_1}h_e g_{v_2}^{-1}$, where e is an edge of Γ connecting the vertices v_1 and v_2.

C. The symplectic structures

In this subsection we use the map (3.24) to prove the equivalence of the symplectic structures on the continuous and discrete spaces $\mathcal{P}_{\Gamma,\Gamma^*}$ and P_Γ. We know that the spaces $\mathcal{P}$ and P_Γ are symplectic manifolds, their symplectic structures being given by (1.5) and (2.4) respectively. Since the space $\mathcal{P}_{\Gamma,\Gamma^*}$ has been obtained from $\mathcal{P}$ by symplectic reduction, the Marsden-Weinstein theorem ensures that it also carries a symplectic structure. We are now going to show that the symplectic structures on the spaces $\mathcal{P}_{\Gamma,\Gamma^*}$ and P_Γ are in fact identical.

Let us start with the symplectic potential coming from the first order formulation of gravity. It is given by

$$\Theta = \frac{1}{2}\int_\Sigma \mathrm{Tr}\left(\star(e\wedge e)\wedge\delta A\right) = \int_\Sigma \mathrm{Tr}\left(E\wedge\delta A\right), \tag{3.30}$$

where $\star$ denotes the Hodge duality map in the Lie algebra $\mathfrak{su}(2)$. We first use the cellular decomposition Δ to evaluate this symplectic potential on the set of partially flat connections and write

$$\begin{aligned}
\Theta &= \sum_v \int_{C_v} \mathrm{Tr}\left(E\wedge\delta\left(a_v \mathrm{d}a_v^{-1}\right)\right) && (3.31\mathrm{a})\\
&= \sum_v \int_{C_v} \mathrm{Tr}\left(X_v\wedge\mathrm{d}\left(\delta a_v^{-1}a_v\right)\right) && (3.31\mathrm{b})\\
&= \sum_v \int_{\partial C_v} \mathrm{Tr}\left(X_v\wedge\delta a_v^{-1}a_v\right) - \sum_v G_v\delta a_v^{-1}a_v(v) && (3.31\mathrm{c})\\
&= \sum_v \int_{\partial C_v} \mathrm{Tr}\left(X_v\wedge\delta a_v^{-1}a_v\right), && (3.31\mathrm{d})
\end{aligned}$$

where we have used the identity $\delta\left(a_v\mathrm{d}a_v^{-1}\right) = a_v\mathrm{d}\left(\delta a_v^{-1}a_v\right)a_v^{-1}$, the definition (3.14) of the two-form field X_v and the fact that $\mathrm{d}X_v = G_v\delta(x-v)$ (see eq.3.26). The last equality follows from the condition $a_v(v)=1$, hence $\delta a_v(v)=0$. The summation over three-cells dual to the vertices v can be rearranged as a sum over two-cells dual to the edges e, which gives

$$\Theta = \sum_e \int_{F_e} \left[\mathrm{Tr}\left(X_{s(e)}\wedge\delta a_{s(e)}^{-1}a_{s(e)}\right) - \mathrm{Tr}\left(X_{t(e)}\wedge\delta a_{t(e)}^{-1}a_{t(e)}\right)\right]. \tag{3.32}$$

Now we can use the condition (3.18) of compatibility of the group elements a_v across the edges to rewrite the second term and obtain

$$\Theta = \sum_e \int_{F_e} \left[\mathrm{Tr}\left(X_{s(e)}\wedge\delta a_{s(e)}^{-1}a_{s(e)}\right) - \mathrm{Tr}\left(h_e^{-1}X_{s(e)}h_e\wedge\delta\left(h_e^{-1}a_{s(e)}^{-1}\right)a_{s(e)}h_e\right)\right]. \tag{3.33}$$

Finally, we can expand the last term to find the result

$$\Theta = -\sum_e \int_{F_e} \mathrm{Tr}\left(h_e^{-1}X_{s(e)}h_e\wedge\delta h_e^{-1}h_e\right) = \sum_e \mathrm{Tr}\left(X_e\delta h_e h_e^{-1}\right). \tag{3.34}$$

This is exactly the symplectic potential associated to $|E_\Gamma|$ copies of the cotangent bundle $T^*\mathrm{SU}(2)$. It shows that the symplectic structure of the spin network phase space is equivalent to that of first order gravity evaluated on the set of partially flat connections. In particular, since the symplectic forms are invertible by definition, this proves that the continuous phase space $\mathcal{P}_{\Gamma,\Gamma^*}$ is indeed

finite-dimensional and isomorphic to P_Γ.

D. Action of diffeomorphisms

Now we prove the second point of proposition (1), which concerns the invariance of the symplectomorphism under a certain class of diffeomorphisms. The isomorphism $\mathcal{I}:\mathcal{P}_{\Gamma,\Gamma^*}\longrightarrow P_\Gamma$ depends on a choice of cellular decomposition Δ dual to Γ with one-skeleton $\Delta_1=\Gamma^*$. Diffeomorphisms $\Phi\in\mathrm{Diff}(\Sigma)$ act naturally on the continuous phase space $\mathcal{P}_{\Gamma,\Gamma^*}$ by $A\longmapsto\Phi^*A$ and $E\longmapsto\Phi^*E$.

Let us start by choosing a particular diffeomorphism Φ_o which preserves the graph Γ^* and the vertices V_Γ inside the cells C_v, and is connected to the identity[4]. Because the connection is flat on $\tilde{\Sigma}$, the holonomy $h_e(A)$ is independent of the choice of path between $s(e)$ and $t(e)$ as long as any two paths are in the same homotopy class of $\tilde{\Sigma}$. The edges e and $\Phi_o(e)$ are in the same homotopy class if Φ_o is connected to the identity and not moving Γ^*. Then it is clear that we have

$$h_e(\Phi_o^*A)=h_{\Phi_o(e)}(A)=h_e(A). \tag{3.35}$$

Similarly, the action of Φ_o on the group element $a_v(x)$ maps it to $a_v\big(\Phi_o(x)\big)$. This implies that the two-form X_v defined by (3.14) satisfies $X_v\big(\Phi_o(x)\big)=\Phi_o^*X_v(x)$. Now, since Φ_o does not move the graph Γ^*, we have that $\partial F_e=\partial\big(\Phi_o(F_e)\big)\in\Gamma^*$, and therefore $F_e\cup\Phi_o(F_e)$ encloses a volume, which furthermore does not contain any vertices of Γ. Thus, by virtue of (2.25) and (3.21), we have that

$$X_e(\Phi_o^*A,\Phi_o^*E)=X_e(A,E). \tag{3.36}$$

Relations (3.35) and (3.36) together show that $\mathcal{I}\circ\Phi_o=\mathcal{I}$.

We can give another very elegant proof of the invariance of the map $\mathcal{I}$ under the diffeomorphisms Φ_o. For this, recall that given a vector field ξ^a, a diffeomorphism acts on the connection like

$$\mathcal{L}_\xi A=\mathrm{d}(\iota_\xi A)+\iota_\xi\mathrm{d}A=\iota_\xi F+\mathrm{d}_A(\iota_\xi A), \tag{3.37}$$

and on the electric field like

$$\mathcal{L}_\xi E=\mathrm{d}(\iota_\xi E)+\iota_\xi\mathrm{d}E+[E,\iota_\xi A]=\iota_\xi\mathrm{d}_AE+\mathrm{d}_A(\iota_\xi E)+[E,\iota_\xi A], \tag{3.38}$$

where ι denotes the interior product. Now, if the data (A,E) is on the constraint surface $\mathcal{C}$, the curvature vanishes outside of Γ^*, while d_AE vanishes outside of the set V_Γ of vertices. Therefore, if we consider a vector field ξ^a which vanishes on Γ^* and on V_Γ, we see that the action of diffeomorphisms is a combination of flat transformations (3.3) and gauge transformations (3.6) with field-dependent parameters of transformation:

$$\mathcal{L}_\xi=\delta^{\mathcal{F}_{\Gamma^*}}_{\iota_\xi E}+\delta^{\mathcal{G}_\Gamma}_{\iota_\xi A}. \tag{3.39}$$

Therefore, such diffeomorphisms vanish on the gauge-invariant variables (h_e,X_e).

[4] This means that there exists a smooth one-parameter family of diffeomorphism Φ_t such that $\Phi_{t=0}=\mathrm{id}$ and $\Phi_{t=1}=\Phi_o$.

IV. GAUGE CHOICES FOR THE ELECTRIC FIELD

Now that we have established the isomorphism between P_Γ and the continuous phase space $\mathcal{P}_{\Gamma,\Gamma^*}$, we have a correspondence between discrete geometries and an equivalence class of continuous geometries related according to (3.27) by group gauge transformations and translations. Up to group gauge transformations, the holonomy uniquely determines a choice of connection. For the E field however, the story is different since even after we perform a group gauge transformation, there is still a huge ambiguity $E \longmapsto E + \mathrm{d}_A\phi$ on the continuous electric field determined by the fluxes. It is clear that in order to construct a continuous field configuration starting from the discrete data, one has to specify which continuous field representative to pick in the particular equivalence class determined by the discrete data. In other words, a choice of a representative in this equivalence class is a choice of gauge. More precisely, we have the following definition:

Definition 4. *A choice of gauge is a map from the discrete data to the continuous phase space,*

$$\begin{array}{rccc} \mathcal{T}: & P_\Gamma & \longrightarrow & \mathcal{C}_{\Gamma,\Gamma^*} \\ & (h_e, X_e) & \longmapsto & (A, E), \end{array} \tag{4.1}$$

which is the inverse of $\mathcal{I}$ in the sense that

$$\mathcal{I} \circ \mathcal{T} = \mathrm{id}. \tag{4.2}$$

We say that a gauge fixing is diffeomorphism-covariant if $\Phi^*\mathcal{T}$ is equal to the map $\mathcal{T}$ defined on the graphs $\Phi(\Gamma)$ and $\Phi(\Gamma^*)$, for any diffeomorphism $\Phi : \Sigma \longrightarrow \Sigma$.

In other words, choosing a gauge amounts to giving a prescription for reconstructing continuous fields $A(h_e)$ and $E(X_e, h_e)$ starting from the discrete data, such that (4.2) holds, i.e.

$$h_e\big(A(h_e)\big) = h_e, \qquad\qquad X_e\big(A(h_e), E(X_e, h_e)\big) = X_e. \tag{4.3}$$

Note that a gauge fixing $\mathcal{T}$ is a right inverse for $\mathcal{I}$, while the reverse is not true. The map $\mathcal{T} \circ \mathcal{I}$ is not the identity, it just maps a continuous configuration (A, E) that solves the Gauss and curvature constraints into another gauge-equivalent configuration which satisfies the gauge choice.

As we have already seen, at the continuous level a flat connection on $\tilde{\Sigma}$ is determined on every cell C_v by a group element $a_v(x)$. Locally, it is always possible to perform a gauge transformation that sends this element to the identity of the group, and thereby construct a trivial connection. If we pick two neighboring cells C_{v_1} and C_{v_2} such that the vertices v_1 and v_2 bound the edge dual to the face $F_e = \overline{C_{v_1}} \cap \overline{C_{v_2}}$, the relevant gauge-invariant information about the connection is encoded in the transition group element h_e.

For the electric field, there is more gauge freedom since the variable E can be acted upon by both $\mathcal{F}_{\Gamma^*}$ and $\mathcal{G}_\Gamma$. Therefore, there is a priori a huge ambiguity in the choice of gauges that one can choose to reconstruct the continuous data. This means that knowledge of the fluxes does not accurately determine the geometry of space, but only a family of geometries that are gauge-equivalent under translations of the type $E \longmapsto E + \mathrm{d}_A\phi$.

However, there is a powerful way in which we can restrict the gauge choices that are available. This can be done by asking that a gauge choice transforms covariantly under the action of diffeomorphisms. A diffeomorphism Φ of Σ acts on the continuous data in the usual manner $(A, E) \longmapsto \big(\Phi^*A, \Phi^*E\big)$. The same diffeomorphism also acts on the discrete data (h_e, X_{F_e}) as $\big(h_{\Phi(e)}, X_{\Phi(F_e)}\big)$. Note that here we have made explicit the fact that the flux field X_e depends on Γ^* via the choice of a surface F_e whose boundary is supported on Γ^*. A gauge choice is said to be covariant if this action of the diffeomorphisms commutes with the gauge map $\mathcal{T}$.

If we restrict ourselves to gauge choices that are covariant under the action of diffeomorphisms, the ambiguity in the gauge choices is dramatically resolved, and there are only a few choices available. In the following we present two such gauge choices[5]. First, the singular gauge choice in which the electric field E vanishes outside of Γ, and then the flat gauge in which E is flat outside of Γ^*. It is remarkable that these two gauge choices correspond to the two main interpretations of the fluxes used in the literature. In loop quantum gravity one usually interprets the E field as having support only on Γ, whereas in the spin foam literature one usually interprets the E field as being flat outside of Γ^*. Our analysis shows that these two pictures are not contradictory, but that they correspond to two different covariant gauge choices underlying the same discrete data.

Now we want to emphasize that the restriction on the gauge choices coming from the requirement of covariance under diffeomorphisms is the analog of the so-called uniqueness theorem of the quantum representation of the holonomy-flux algebra [24]. This theorem states that there is a unique diffeomorphism-covariant gauge choice, which corresponds to the singular gauge in which E has support on the graph Γ only and vanishes on Γ^*. In this singular gauge, which we refer to as the LQG gauge, the electric field E vanishes outside of the graph Γ dual to the triangulation Δ. This can be written as $E|0\rangle = 0$, where the vacuum state $|0\rangle$ is the state of no geometry. Indeed, in LQG excitations of quantum geometry have support on the graph Γ only. Therefore, in all the regions of Δ outside of Γ, there is simply no geometry, and the electric field vanishes. We are going to give below an explicit construction of the continuous singular electric field.

The key observation is that there is another legitimate choice of representative configuration in the equivalence class (3.27) of continuous geometries which respects the diffeomorphism symmetry. As we already said, it is given by the flat gauge. At the quantum level, this corresponds to a choice of a vacuum state $|0_{\mathrm{F}}\rangle$ in which the curvature vanishes. This corresponds to the flat, or spin foam gauge, in which we have $F(A)|0_{\mathrm{F}}\rangle = 0$. This diffeomorphism-invariant vacuum is missed by the LOST theorem due to additional technical hypotheses. It is interesting to note that such a vacuum state appears naturally in our context and that it corresponds to the spin foam description. It can be seen as the dual of the singular gauge, in the sense that it defines a flat geometry within the cells C_v, with a non-vanishing electric field E on the dual graph Γ^*. As we will see in more detail, the availability of this gauge clearly shows that it is possible to define a locally flat geometry without necessarily having a triangulation with straight edges and flat faces. In Regge geometries [6], the extrinsic curvature is concentrated along the one-skeleton Δ_1 of the triangulation, but in the present construction, the edges of Γ^* are not necessarily straight.

Here we have drawn a parallel between a choice of gauge at the classical level and a choice of a vacuum state at the quantum level. It would be interesting to develop this analogy further.

Before giving more details about the gauge choices for the electric field, let us make a comment about the gauge group $\mathcal{F}_{\Gamma^*}$. We have seen previously that the flatness constraint generates a transformation of the electric field given by

$$\delta_\phi^{\mathcal{F}_{\Gamma^*}} E = \mathrm{d}_A \phi. \tag{4.4}$$

Since we have constructed the gauge group $\mathcal{F}_{\Gamma^*}$ with the condition $\phi(x) = 0$ for $x \in \Gamma^*$, one may question whether this imposes a restriction on the gauge choices that we can obtain using the transformation (4.4). The following lemma ensures there is no such restriction.

Lemma 1. *Let $\rho \in \Omega^2\big(\Sigma, \mathfrak{su}(2)\big)$ be a Lie algebra-valued two-form on Σ. It is always possible to find a Lie algebra-valued one-form $\phi \in \Omega^1\big(\Sigma, \mathfrak{su}(2)\big)$ vanishing on Γ^* such that $\rho = \mathrm{d}_A\phi$ on Γ^*.*

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[5] We conjecture these are the only two possible gauge choices, but a detailed investigation of this is still needed.

Proof. Consider an element $\rho \in \Omega^2\big(\Sigma, \mathfrak{su}(2)\big)$, along with an edge e^* in Γ^* parametrized by a coordinate x. In a neighborhood of the edge, we can find coordinates y and z which are perpendicular to x. Using $\phi(e^*) = 0$ and therefore $\partial_x\phi|_{e^*} = 0$, we can compute

$$\begin{aligned} \rho(e^*) &= (\mathrm{d}_A\phi)(e^*) = (\mathrm{d}\phi)(e^*) \qquad (4.5a)\\ &= -(\partial_y\phi_x)(e^*)\mathrm{d}x \wedge \mathrm{d}y - (\partial_z\phi_x)(e^*)\mathrm{d}x \wedge \mathrm{d}z + (\partial_y\phi_z - \partial_z\phi_y)(e^*)\mathrm{d}y \wedge \mathrm{d}z. \qquad (4.5b)\end{aligned}$$

Choosing $\phi_z = 0$, we can find a solution in the neighborhood of e^*, given by

$$\phi_x = -\rho_1 y - \rho_2 z, \qquad \phi_y = -\rho_3 z, \qquad \phi_z = 0, \tag{4.6}$$

where the constants ρ_1, ρ_2 and ρ_3, are the components of ρ evaluated on the edge, i.e.

$$\rho_1 = \rho_{xy}(e^*), \qquad \rho_2 = \rho_{xz}(e^*), \qquad \rho_3 = \rho_{yz}(e^*). \tag{4.7}$$

There are of course other possible solutions, but we only give one to establish the proof. ■

In the remainder of this section, we are going to study in more detail the singular and flat gauges for the electric field. Our goal is to study the gauge freedom for the basic variables on the continuous phase space, and to construct explicitly the electric field as a functional of the discrete variables h_e and X_e.

A. Singular gauge

The singular gauge is a gauge in which the electric field E vanishes outside of the graph Γ. In this section, we show by an explicit construction that it always possible to make such a gauge choice. More precisely, we construct explicitly continuous fields $A(h_e)$ and $E_{\mathrm{S}}(h_e, X_e)$ which are such that $E_{\mathrm{S}}(x) = 0$ if $x \notin \Gamma$, and which satisfy the property $\mathcal{I}(A, E_{\mathrm{S}}) = (h_e, X_e)$ under the action of the map (3.24).

In order to prove this, let us first introduce the following form:

$$\omega(x,y) \equiv \omega^i(x-y)\epsilon_{ijk}\mathrm{d}x^j \wedge \mathrm{d}y^k, \qquad \text{with} \qquad \omega^i(x) \equiv \frac{1}{4\pi}\frac{x^i}{|x|^3}. \tag{4.8}$$

This object is a (1,1)-form, i.e. a one-form in x, and a one-form in y. This form satisfies a key property, which is summarized in the following lemma.

Lemma 2. *There exists an $\alpha(x,y)$ which is a (2,0)-form (i.e. a two-form in x and a zero-form in y), such that*

$$\mathrm{d}_x\omega(x,y) + \mathrm{d}_y\alpha(x,y) = \delta(x,y), \tag{4.9}$$

where $\mathrm{d}_x \equiv \mathrm{d}x^i\partial_{x^i}$, and $\delta(x,y)$ is the distributional (2,1)-form

$$\delta(x,y) = \delta(x-y)\epsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}y^k \tag{4.10}$$

vanishing outside of $x = y$.

Proof. First, it is straightforward to show that $\partial_i\omega^i(x) = 0$ for $x \neq 0$. Moreover, it is possible to show by a direct computation in spherical coordinates that

$$\int_{S_\varepsilon} \omega^i(x)\epsilon_{ijk}\mathrm{d}x^j \wedge \mathrm{d}x^k = 2, \tag{4.11}$$

where S_ε is a sphere of radius ε. Since this integral is also equal to

$$2\int_{B_\varepsilon} \partial_i \omega^i(x) \mathrm{d}^3 x, \tag{4.12}$$

where B_ε is the ball of radius ε, we obtain that $\partial_i \omega^i(x) = \delta(x)$. By a direct computation we can now get that

$$\mathrm{d}_x \omega(x,y) = s_k \wedge \mathrm{d}y^k \left(\partial_i \omega^i\right)(x-y) - s_j \wedge \mathrm{d}y^k \left(\partial_k \omega^j\right)(x-y), \tag{4.13}$$

with

$$s_i = \frac{1}{8\pi} \epsilon_{ijk} \mathrm{d}x^j \wedge \mathrm{d}x^k. \tag{4.14}$$

The lemma is therefore established by introducing $\alpha(x,y) \equiv \omega^i(x-y) s_i$. ■

Given this lemma, it is now a straightforward task to construct a singular flux field. For this, we first construct a flat connection A on $\tilde{\Sigma}$ following the construction of subsection III A, and then we define the singular flux field as

$$E_{\mathrm{S}}(x) \equiv \mathrm{d}_A \left(\sum_{e \in \Gamma} h_{\pi_e}(x)^{-1} X_e h_{\pi_e}(x) \int_{e(y)} \omega(x,y) \right). \tag{4.15}$$

The integral entering this definition is a one-dimensional integral over the edge e parametrized by the variable y, which implies that the term inside the parenthesis is a one-form in x.

The proof that this flux satisfies all the desired requirements is straightforward. First, it is obvious that the Gauss law $\mathrm{d}_A E_{\mathrm{S}} = 0$ is satisfied on $\Sigma \backslash \Gamma^*$ since $\mathrm{d}_A^2 = F(A) = 0$ on this space. Moreover, using the previous lemma and the definition of the holonomy, we can compute explicitly the covariant derivative:

$$E_{\mathrm{S}}(x) = \sum_e h_{\pi_e}(x)^{-1} X_e h_{\pi_e}(x) \Big(\delta_e(x) - \alpha\big(x, s(e)\big) + \alpha\big(x, t(e)\big) \Big), \tag{4.16}$$

where

$$\delta_e(x) \equiv \int_{e(y)} \delta(x,y). \tag{4.17}$$

The last two terms in (4.16) can be reorganized in terms associated with the vertices to find

$$E_{\mathrm{S}}(x) = \sum_e h_{\pi_e}(x)^{-1} X_e h_{\pi_e}(x) \delta_e(x) - \sum_v \alpha(x,v) h_v(x)^{-1} \left(\sum_{e|s(e)=v} X_e - \sum_{e|t(e)=v} h_e^{-1} X_e h_e \right) h_v(x), \tag{4.18}$$

where $h_v(x)$ is the holonomy going from the vertex v to the point x. Now the last term vanishes due to the discrete Gauss law (2.6). Therefore, we finally find that the singular electric field is

$$E_{\mathrm{S}}(x) = \sum_e h_{\pi_e}(x)^{-1} X_e h_{\pi_e}(x) \delta_e(x). \tag{4.19}$$

This electric field is obviously vanishing outside of Γ, and is such that $X_e(A, E_{\mathrm{S}}) = X_e$. It is

interesting to note that the integral of the two-form $\alpha(x,y)$ along S,

$$\int_S \alpha(x,y) = \frac{1}{8\pi}\int_S \omega^i(x-y)\epsilon_{ijk}\mathrm{d}x^j \wedge \mathrm{d}x^k, \tag{4.20}$$

is simply the solid angle of S as viewed from y divided by 4π.

B. Flat cell gauge

In this subsection, we prove that it is always possible to perform a gauge transformation which leads to a flat geometry within each cell. Since the connection A is flat on $\tilde{\Sigma}$, a flat geometry on $\tilde{\Sigma}$ is determined by the choice of a frame field (i.e. an invertible $\mathfrak{su}(2)$-valued one-form $e = e^i\tau_i$) which satisfies the torsion-free condition $\mathrm{d}_A e = 0$ in $\tilde{\Sigma} = \Sigma\backslash\Gamma^*$. Indeed, since $\mathrm{d}_A e = \gamma[K,e]$, the vanishing of the torsion and the invertibility of e together imply that $K = 0$, and therefore the SU(2) Ashtekar-Barbero connection is simply equal to the spin connection $A = \Gamma(e)$. The flatness of the SU(2) connection therefore imposes the flatness of the spin connection, which implies that the metric determined by e is flat.

To construct such a flat geometry, we need to obtain an electric field E so that $\mathrm{d}_A e = 0$ within each cell C_v. However, E as defined does not necessarily imply this condition. Recall that we have an equivalence class of electric fields yielding the same integrated flux, related by transformations generated by the flatness constraint $\mathcal{F}_{\Gamma^*}$. Given the gauge transformation in (3.3), is there a choice of ϕ which leads to a flat geometry? To answer this question, we need to define the action of the constraint $\mathcal{F}_{\Gamma^*}$ on the triad field. The action on the electric field is given by

$$\left(\mathrm{d}_A\phi^i\right)_{bc} = \left\{E^i_{bc}, \mathcal{F}_{\Gamma^*}(\phi)\right\} = \epsilon^i{}_{jk}\left\{e^j_b e^k_c, \mathcal{F}_{\Gamma^*}(\phi)\right\} = \epsilon^i{}_{jk} e^j_{[b}\delta^{\mathcal{F}_{\Gamma^*}}_{\phi} e^k_{c]}. \tag{4.21}$$

One can show that this leads to the following transformation property for the triad field:

$$\delta^{\mathcal{F}_{\Gamma^*}}_{\phi} e^i_a = \left(\frac{1}{2}e^b_j e^i_a - e_{aj}e^{bi}\right)\left(\star_e \mathrm{d}_A\phi\right)^j_b, \tag{4.22}$$

where $\star_e$ denotes the hodge star operator determined by e. Considering this, let us define a map $M_e : \Omega^1\big(\Sigma,\mathfrak{su}(2)\big) \longrightarrow \Omega^1\big(\Sigma,\mathfrak{su}(2)\big)$, given by

$$M_e(\phi) \equiv \left(\frac{1}{2}e^a_i e^j_b - e_{bi}e^{aj}\right)\left(\star_e \mathrm{d}_A\phi\right)^i_a. \tag{4.23}$$

This map is clearly a homomorphism. Now, since $\left(e^b_j e^i_a/2 - e_{aj}e^{bi}\right)$ is invertible, the kernel of this map is the space $Z^1\big(\Sigma,\mathfrak{su}(2)\big)$ of twisted-closed one-forms, that is, the space of all $\mathfrak{su}(2)$ valued one-forms ω such that $\mathrm{d}_A\omega = 0$. Then, using the fundamental theorem of homomorphisms, we have that

$$\mathrm{im}(M_e) \cong \frac{\Omega^1\big(\Sigma,\mathfrak{su}(2)\big)}{\ker(M_e)} = \frac{\Omega^1\big(\Sigma,\mathfrak{su}(2)\big)}{Z^1\big(\Sigma,\mathfrak{su}(2)\big)}. \tag{4.24}$$

Proof. Let us define a map

$$m_e : \frac{\Omega^1\big(\Sigma,\mathfrak{su}(2)\big)}{Z^1\big(\Sigma,\mathfrak{su}(2)\big)} \longrightarrow \mathrm{im}(M_e) \tag{4.25}$$

by $m_e([\phi]) = M_e(\phi)$. This map is well defined, since for some $\alpha \in Z^1\big(\Sigma, \mathfrak{su}(2)\big)$, we can write $[\phi] \ni \phi' = \phi + \alpha$, and ϕ and ϕ' are mapped to the same point:

$$M_e(\phi') = M_e(\phi + \alpha) = M_e(\phi) + M_e(\alpha) = M_e(\phi). \tag{4.26}$$

The map m_e is one-to-one, since we can write

$$\begin{aligned} m_e([\phi_1]) &= m_e([\phi_2]) \\ \Longleftrightarrow \quad M_e(\phi_1 + \alpha_1) &= M_e(\phi_2 + \alpha_2) \\ \Longleftrightarrow \quad M_e(\phi_1) &= M_e(\phi_2) \\ \Longleftrightarrow \quad M_e(\phi_1 - \phi_2) &= 0, \end{aligned} \tag{4.27}$$

which shows that $\phi_1 - \phi_2 \in \ker(M_e)$ and $[\phi_1] = [\phi_2]$. Finally, the map m_e is onto, since for every $\psi \in \mathrm{im}(M_e)$ there exists a ϕ in the target space such that $M_e(\phi) = \psi = m_e([\phi])$. ■

New let us go back to our problem, and suppose that we have an electric field E which does not define a flat geometry. A non-flat triad satisfies $\mathrm{d}_A e \neq 0$, which implies that

$$e \in \frac{\Omega^1\big(\Sigma, \mathfrak{su}(2)\big)}{Z^1\big(\Sigma, \mathfrak{su}(2)\big)} \cong \mathrm{im}(M_e). \tag{4.28}$$

This means that within C_v, we can always choose a ϕ such that $M_e(\phi) = -e + \beta$ where $\beta \in Z^1\big(\Sigma, \mathfrak{su}(2)\big)$, and therefore obtain

$$\mathrm{d}_A\big(e + M_e(\phi)\big) = 0, \tag{4.29}$$

which defines a flat geometry. This shows the existence of a gauge choice with a flat frame field e, which is what we desired. Note that since there is an equivalence class of flat triads related by diffeomorphisms, the choice of ϕ is not unique.

We are now interested in reconstructing the flux elements X_e starting from the flat frame field e. To do so, let us first define the gauge-transformed frame field

$$e_v \equiv h_{\pi_e}(x) e(x) h_{\pi_e}(x)^{-1}, \tag{4.30}$$

where the holonomy starts at the vertex v in the cell (and does not depend on the path). This definition will enable us to define the flux (2.15) with the appropriate smearing. The torsion-free condition $\mathrm{d}_A e = 0$ implies that e_v is twisted-closed and hence twisted-exact on C_v. This means that the frame field can be written as

$$e(x) = a_v(x) \mathrm{d}x_v a_v(x)^{-1}, \tag{4.31}$$

where the zero-form x_v provides a set of flat coordinates on the cell C_v. Then, the covariant discrete flux elements are simply given by

$$X_e^i = \frac{1}{2}\epsilon^i{}_{jk} \int_{F_e} e_v^j \wedge e_v^k = \frac{1}{2} a_v(v) \left(\epsilon^i{}_{jk} \int_{F_e} \mathrm{d}x_v^j \wedge \mathrm{d}x_v^k \right) a_v(v)^{-1} = \frac{1}{2}\epsilon^i{}_{jk} \int_{\partial F_e} x_v^j \mathrm{d}x_v^k, \tag{4.32}$$

where we have used the fact that $a_v(v) = 1$.

C. Regge geometries and cotangent bundle

The previous calculation shows that we can think of the phase space P_Γ as the phase space of piecewise (metric) flat geometries on $\Sigma\backslash\Gamma^*$. Such geometries possess an invertible locally flat metric, with curvature concentrated on the one-skeleton of the cellular decomposition. This description is reminiscent of Regge geometries. However, it is known that the phase space of loop gravity is bigger than the phase space of Regge geometry [7]; Regge geometries appear only as a constrained subset. This fact has triggered the search for the proper geometrical interpretation of the loop gravity phase space, for instance in terms of twisted geometries [5]. We can now clearly understand the key difference between the phase space of loop gravity and that of Regge geometries. The loop gravity phase space corresponds to piecewise flat geometries on $\Sigma\backslash\Gamma^*$ while the Regge phase space corresponds to piecewise-linear flat geometries on $\Sigma\backslash\Gamma^*$. This means that the geometry is flat, but also that the edges of Γ^* and the faces of the two-complex are flat lines and planes. It is this additional restriction which allows us to identify a loop gravity configuration with a Regge configuration.

To understand how this comes about, let us go back to the formula for the fluxes that we have derived in the previous subsection:

$$X_e^i = \frac{1}{2}\epsilon^i{}_{jk}\int_{F_e} \mathrm{d}x_v^j \wedge \mathrm{d}x_v^k, \tag{4.33}$$

where x_v is the flat coordinate in the cell C_v. One sees that if the two-cells are chosen to be flat, then $\mathrm{d}x_v^k$ is constant over F_e and the expression simplifies drastically since the fluxes can be expressed as a cross product of discrete frame fields. This condition implies that the fluxes can be constructed entirely in terms of a discrete piecewise flat geometry à la Regge, and that they satisfy the so-called gluing constraints [7]. This means that a set of fluxes satisfying the gluing constraints corresponds to a Regge geometry and can be implemented as a piecewise flat geometry on $\Sigma\backslash\Gamma^*$ with the additional constraint that the edges of Γ^* are straight with respect to the flat structure[6]. The phase space of full loop gravity then corresponds to piecewise geometries where this additional restriction is not imposed. In other words, the edges of Γ^* do not have to be flat even if the curvature is concentrated on them. Our construction shows that this additional restriction is not necessary.

The meaning of the additional Regge restriction becomes even clearer when expressed in terms of the extrinsic curvature. What happens in Regge geometry is that both the intrinsic and extrinsic geometry are concentrated on the flat edge. However, the extrinsic geometry K_a^i is not allowed to freely fluctuate since the condition of flatness of the edge amounts to demanding that the extrinsic curvature has non-zero components only in the direction parallel to the edge, i.e. $K_a^i = \dot{e}_a K^i$ where $\dot{e}_a$ is the vector tangent to the edge. The Gauss law further imposes that K^i is also parallel to the edge. Therefore, in Regge geometry, we can access (up to a rotation) only one component of the extrinsic curvature tensor (the deficit angle). This unnecessary restriction is relaxed in the loop gravity phase space, since the extrinsic curvature tensor is now allowed to freely fluctuate as it should. From this point of view, we see that the phase space of loop gravity possesses extra freedom which allows us to fully capture the dynamics. We hope to come back to these points in the future.

The result of our construction is that after a choice of gauge, we can express the elements of P_Γ

[6] This means that $\mathrm{d}x_v$ is constant on the edges of Γ^*.

as a connection A and an $\mathfrak{su}(2)$-valued frame field one-form e, which are solutions to

$$F(A)(x) = 0, \qquad \mathrm{d}_A e(x) = 0, \qquad \forall\, x \in \tilde{\Sigma} = \Sigma \backslash \Gamma^*. \tag{4.34}$$

Since $\delta F(A) = \mathrm{d}_A \delta A$, this is nothing but the cotangent bundle of the space of flat SU(2) connections on $\Sigma\backslash\Gamma^*$. That is

$$P_\Gamma = T^* \mathcal{M}_{\Gamma^*}, \tag{4.35}$$

where $\mathcal{M}_{\Gamma^*}$ denotes the moduli space of flat connections modulo gauge transformations. This means that at the quantum level we can represent the quantization of holonomies and fluxes in terms of operators acting on holonomies of flat connections. This interpretation has already proposed by Bianchi in [8]. It is interesting to note that this is reminiscent of the geometry considered by Hitchin in [25].

D. Diffeomorphisms and gauge choices

We have seen in subsection III D that diffeomorphisms Φ_o connected to the identity that do not move Γ^* and the vertices of Γ leave the construction of the holonomy-flux algebra invariant. We have also seen in the beginning of this section that the singular gauge and the flat gauge are diffeomorphism covariant. In general, the construction of h_e and X_{F_e} depends both on Γ via the choice of e, and on Γ^* via the choice of a two-cell F_e. Now, because of the flatness of the connection, the holonomy does not really depend on the choice of edge e, but solely on the choice of the homotopy class of e, which itself is left unchanged by diffeomorphisms that are connected to the identity. It is interesting to note that the choice of the singular gauge is invariant under a diffeomorphism that does not move Γ, whereas the choice of the flat gauge is invariant under diffeomorphisms that do not move Γ^*. Indeed, in the singular gauge the frame field depends on the choice of an edge $e \in \Gamma$, and we have $\Phi^* E = E$ if $\Phi(\Gamma) = \Gamma$. Moreover, under an infinitesimal diffeomorphism ξ, the flux becomes

$$\delta_\xi X_e = \int_{\partial F_e} \iota_\xi \left(h_{\pi_e}(x) E(x) h_{\pi_e}(x)^{-1} \right), \tag{4.36}$$

where $h_{\pi_e}(x)$ is again the holonomy going from the source vertex of the edge e to the point x in F_e. We clearly see that this expression vanishes for all ξ when the electric field is in the singular gauge. In the flat gauge, the flux does not depend on Γ, and the construction is therefore invariant under diffeomorphisms leaving Γ^* invariant.

This shows that there is an interesting duality between the two gauges. While the singular gauge respects diffeomorphism invariance with respect to Γ, the flat one respects diffeomorphism invariance with respect to Γ^*.

V. CYLINDRICAL CONSISTENCY

In this section, we analyze to what extent the knowledge of a collection of operators on P_Γ for all Γ determines a continuous operator. Given a collection of operators $\mathcal{O}_\Gamma \in P_\Gamma$, we introduce the notion of cylindrical consistency:

Definition 5. *Suppose that we are given a collection of operators $\mathcal{O}_\Gamma \in P_\Gamma$. We say that such a collection of operators is cylindrically consistent if there exists a continuous operator $\mathcal{O}(A, E)$ such*

that its restriction on the constraint surface $\mathcal{C}$ is equal to $\mathcal{O}_\Gamma$. That is

$$\mathcal{O}|_{\mathcal{C}}(A,E) = \mathcal{O}_\Gamma\big(h_e(A), X_e(A,E)\big). \tag{5.1}$$

The results presented in the previous sections show that such a continuous operator $\mathcal{O}(A,E)$ is characterized by the the following property:

Proposition 2. *$\mathcal{O}(A,E)$ is a cylindrical operator if and only if its restriction to the constraint surface $\mathcal{C}$ is invariant under the gauge group $\mathcal{F}_{\Gamma^*} \times \mathcal{G}_\Gamma$ for every pair of dual graphs (Γ, Γ^*).*

Indeed, suppose that we have a functional $\mathcal{O}(A,E)$ defined on the phase space $\mathcal{P}$ such that its restriction to the constraint surface $\mathcal{C}$ is then $\mathcal{O}|_{\mathcal{C}}(A,E)$, where the field configurations now satisfy $F(A) = 0$ outside of the dual graph Γ^*, and $\mathrm{d}_A E = 0$ outside of the vertices V_Γ. $\mathcal{O}$ is a cylindrically consistent operator if and only if

$$\mathcal{O}|_{\mathcal{C}}\big(g \triangleright A, (\phi, g) \triangleright E\big) = \mathcal{O}|_{\mathcal{C}}(A,E), \tag{5.2}$$

which necessarily implies that $\mathcal{O}(A,E) = \mathcal{O}\big(h_e(A), X_e(A,E)\big)$.

This proposition gives us a powerful criterion to check wether a continuous operator can be represented as a collection of operators associated with P_Γ. For instance, we can analyze the status of geometrical operators such as area and volume. We know that the continuous expression for the area operator is

$$\mathbf{A}(S) = \int_S \sqrt{\tilde{E}^i_a \tilde{E}^a_i}. \tag{5.3}$$

One can easily see that even when we restrict this operator to the constraint surface $F(A) = 0$ outside Γ^* and $\mathrm{d}_A E = 0$, this operator is *not* invariant under the translations $E \longmapsto E + \mathrm{d}_A \phi$. Therefore, this operator is *not* expressible purely in terms of holonomies and fluxes associated with the graph Γ. However, in loop quantum gravity, the area operator is expressed as an operator acting on the graph Γ purely in terms of the fluxes:

$$\mathbf{A}_{\mathrm{LQG}}(S) = \sum_{e|e\cap S \neq 0} \sqrt{X^i_e X_{ei}}. \tag{5.4}$$

Our proposition therefore shows that the LQG area operator does not come from the continuous area operator. This means that we have

$$\mathbf{A}(S)|_{\mathcal{C}} - \mathbf{A}_{\mathrm{LQG}}(S) \neq 0. \tag{5.5}$$

So in that sense, the LQG operator is not a proper approximation of the continuous area operator.

This is puzzling since the LQG area operator has been used extensively and derived in many ways. This result thus raises the question of the exact relationship between these two operators. To what extend does the LQG operator capture information about the continuous area operator? Now that we have the exact relationship between the discrete and continuous phase spaces, we can investigate this question a bit further.

First, let us recall that the continuous and LQG area operators are not unrelated. In fact, for any product h_Γ of holonomies supported on the graph Γ, they satisfy

$$\big\{\mathbf{A}(S)|_{\mathcal{C}} - \mathbf{A}_{\mathrm{LQG}}(S), h_\Gamma\big\} = 0. \tag{5.6}$$

So even if $\mathbf{A}|_{\mathcal{C}} - \mathbf{A}_{\mathrm{LQG}}$ does not vanish, it belongs to the commutant of the holonomy algebra.

The second key remark is that if we have a non gauge-invariant operator like $\mathbf{A}(S)$, we can promote it to a gauge-invariant operator under $\mathcal{F}_{\Gamma^*}$ by picking up a gauge. This can be done by working with $\mathbf{A}^{\mathcal{T}}(S) \equiv \mathbf{A}(S)\big(E(X_e)\big)$ instead of $\mathbf{A}(S)(E)$, where $\mathcal{T}$ is a gauge choice as described in section IV. Such an operator is by construction invariant under $\mathcal{F}_{\Gamma^*}$, since it depends only on the fluxes. Moreover, the difference between two operators that differ by a choice of gauge belongs to the commutant of the holonomy algebra:

$$\left\{\mathbf{A}^{\mathcal{T}}(S) - \mathbf{A}^{\mathcal{T}'}(S), h_\Gamma\right\} = 0. \tag{5.7}$$

The relationship between the LQG area operator and the continuous operator is now clear: The LQG area operator is the continuous area operator in the singular gauge. This explains why it can be expressed purely in terms of fluxes. What is less clear is to what extent the knowledge of an operator in a given gauge allows to reconstruct the continuous operator. It is also clear that if one chooses another gauge, like the flat gauge of spin foam models, we are going to construct a different family of area operators associated with graphs, which will differ from $\mathbf{A}_{\mathrm{LQG}}$ by an element of the commutant of the holonomy algebra. It is also not clear which family of operators (if any) we should use to capture in the most efficient way information about the continuous volume operator.

Discussion and conclusion

In this paper, we have shown that the discrete phase space of loop gravity associated with a graph Γ can be interpreted as the symplectic reduction of the continuous phase space of gravity with respect to a constraint imposing the flatness of the connection everywhere outside of the dual graph Γ^*. This allows us to give a clear interpretation of the discrete flux variables as labeling an equivalence class of continuous geometries. The point of view that the discrete data represents a set of continuous geometries has already been advocated in [9]. Our approach gives a precise understanding of which set or equivalence class of continuous geometries is represented by the discrete geometrical data (h_e, X_e) on a graph. It provides a classical understanding of the work by Bianchi [8], who showed that the spin network states can be understood as states of a topological field theory living on the complement of the dual graph. It also allows us to reconcile the tension between the loop quantum gravity picture, in which geometry is thought to be singular, and the spin foam picture, in which the geometry is understood as being locally flat. We now see that both interpretations are valid and correspond to different gauge choices in the equivalence class of geometries represented by the fluxes. It gives us a new understanding of the geometrical operators used in loop quantum gravity as gauged fixed operators, and allows us to investigate further the relationship between these operators and the continuous ones. Finally, it opens the way to a classical formulation of loop gravity. We can now face the question of whether the dynamics of classical general relativity can be formulated in terms of these variables. We plan to come back to this issue of defining a loop classical gravity in the future.

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